

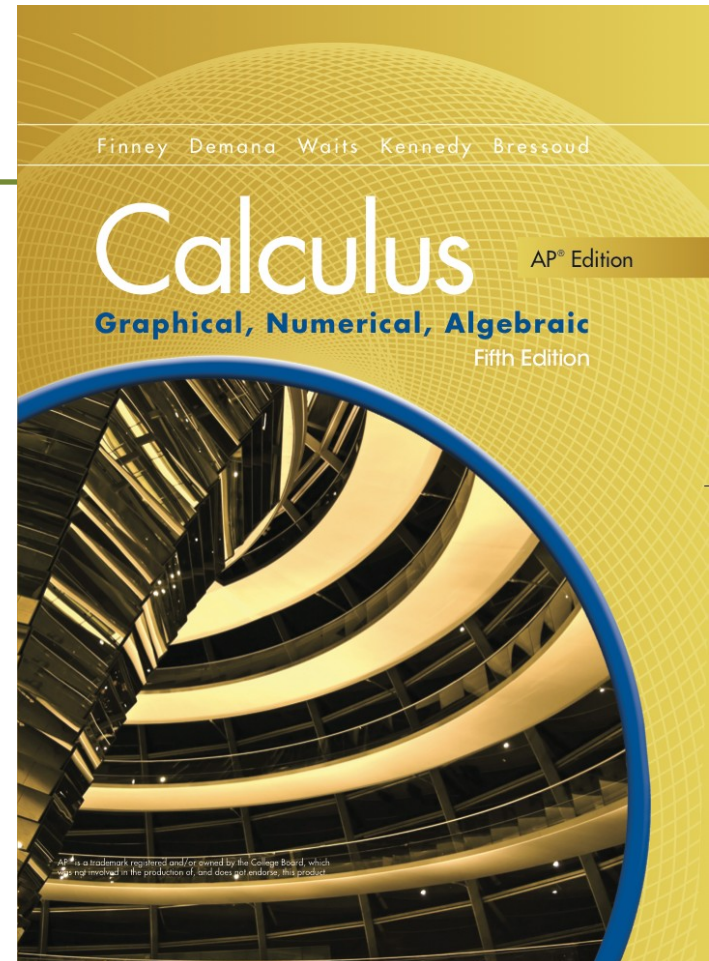
Chapter 7

Differential Equations and Mathematical Modeling



Section 7.1

Slope Fields and Euler's Method



Quick Review

Determine whether or not the function y satisfies the differential equation.

1. $\frac{dy}{dx} = 3y$ $y = e^{3x}$

2. $\frac{dy}{dx} = -\csc^2 x + 5$ $y = \cot x + 5x + 3$

3. $\frac{dy}{dx} = \frac{1}{y}$ $y = \sqrt[3]{3x}$

Quick Review Solutions

Determine whether or not the function y satisfies the differential equation.

1. $\frac{dy}{dx} = 3y$ $y = e^{3x}$ Yes

2. $\frac{dy}{dx} = -\csc^2 x + 5$ $y = \cot x + 5x + 3$ Yes

3. $\frac{dy}{dx} = \frac{1}{y}$ $y = \sqrt[3]{3x}$ No

Quick Review

Find the constant C .

4. $y = 4x^4 + 2x + C$ and $y = 5$ when $x = -1$

5. $y = e^x + \tan x + C$ and $y = -2$ when $x = 0$

Quick Review Solutions

Find the constant C .

4. $y = 4x^4 + 2x + C$ and $y = 5$ when $x = -1$

Let $y = 5$ and $x = -1$

$$5 = 4(-1)^4 + 2(-1) + C$$

$$5 = 4 - 2 + C$$

$$C = 3$$

5. $y = e^x + \tan x + C$ and $y = -2$ when $x = 0$

Let $y = -2$ and $x = 0$

$$-2 = e^0 + \tan(0) + C$$

$$-2 = 1 + C$$

$$C = -3$$

What you'll learn about

- Differential equations
- General and particular solutions of differential equations
- Solving exact differential equations
- Slope fields
- Euler's Method

... and why

Differential equations have been a prime motivation for the study of calculus and remain so to this day.

Differential Equation

An equation involving a derivative is called a **differential equation**. The **order of a differential equation** is the order of the highest derivative involved in the equation.

Example Solving a Differential Equation

Find all functions y that satisfy $\frac{dy}{dx} = 3x^2 + \cos x$

The solution can be any antiderivative of $3x^2 + \cos x$, which can be any function of the form $y = x^3 + \sin x + C$.

First-order Differential Equation

If the general solution to a first-order differential equation is continuous, the only additional information needed to find a unique solution is the value of the function at a single point, called an **initial condition**. A differential equation with an initial condition is called an **initial-value problem**. It has a unique solution, called the **particular solution** to the differential equation.

Example Solving an Initial Value Problem

Find the particular solution to the equation $\frac{dy}{dx} = e^{2x} - 3x$ whose graph passes through the point $\left(1, \frac{1}{2}\right)$.

The general solution is $y = \frac{1}{2} e^{2x} - \frac{3}{2} x^2 + C$.

Applying the initial condition, we have $\frac{1}{2} = \frac{1}{2} e^2 - \frac{3}{2} + C$,

from which we conclude that

$C = 2 - \frac{1}{2} e^2$. Therefore, the particular equation is $y = \frac{1}{2} e^{2x} - \frac{3}{2} x^2 + 2 - \frac{1}{2} e^2$.

Example Using the Fundamental Theorem to Solve an Initial Value Problem

Find the solution to the differential equation

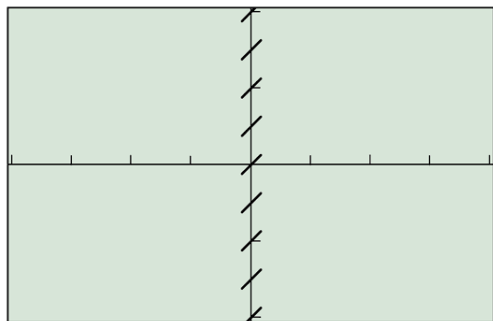
$$f'(x) = \cos(x^2) \text{ for which } f(3) = 5.$$

The solution is $f(x) = \int_3^x \cos(t^2) dt + 5$.

Slope Field

The differential equation gives the slope at any point (x, y) . This information can be used to draw a small piece of the linearization at that point, which approximates the solution curve that passes through that point. Repeating that process at many points yields an approximation called a **slope field**.

Example Constructing a Slope Field



$[-2\pi, 2\pi]$ by $[-4, 4]$

(a)

Construct a slope field for the differential equation

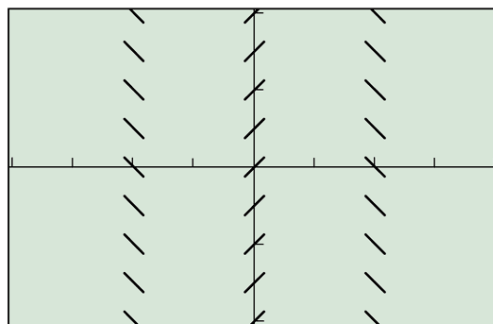
$$\frac{dy}{dx} = \cos x$$

The slope at any point $(0, y)$ will be $\cos 0 = 1$.

The slope at any point (π, y) or $(-\pi, y)$ will be -1 .

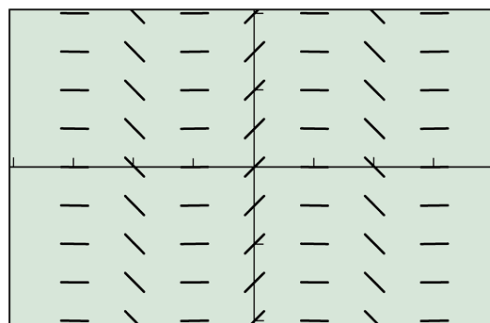
The slope at all odd multiples of $\frac{\pi}{2}$ will be 0.

The slope is 1 along the lines $x = \pm 2\pi$.



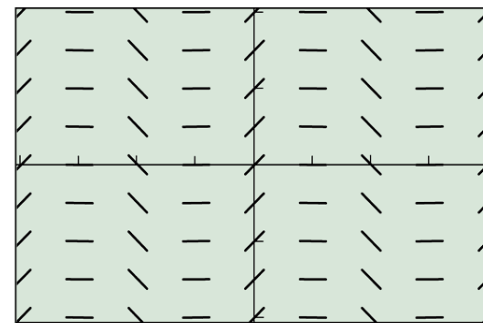
$[-2\pi, 2\pi]$ by $[-4, 4]$

(b)



$[-2\pi, 2\pi]$ by $[-4, 4]$

(c)

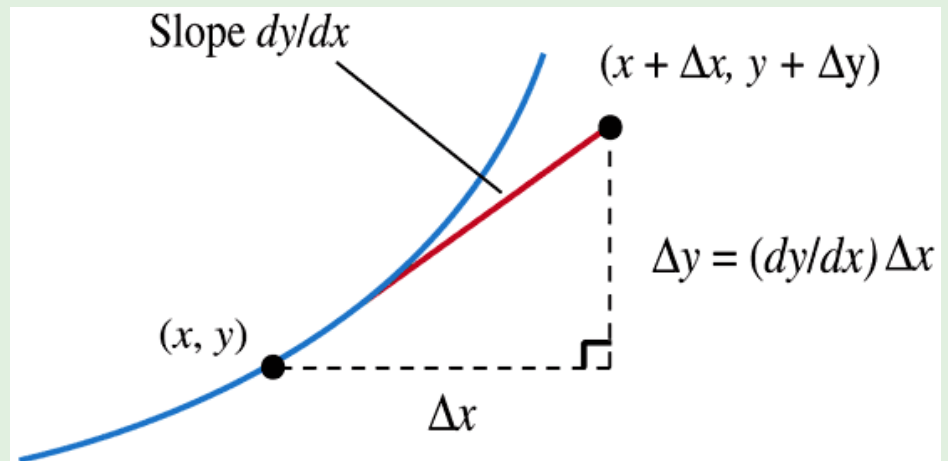


$[-2\pi, 2\pi]$ by $[-4, 4]$

(d)

Euler's Method for Graphing a Solution to an Initial Value Problem

1. Begin at the point (x, y) specified by the initial condition.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by Δx . Increase y by Δy , where $\Delta y = (dy/dx)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization.
4. Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.
5. To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx .



Example Applying Euler's Method

Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.5$ given $\frac{dy}{dx} = x + 1$ and $y = 2$ when $x = 1$.

(x, y)	$\frac{dy}{dx} = x + 1$	Δx	$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$	$(x + \Delta x, y + \Delta y)$
(1, 2)	2	0.1	0.2	(1.1, 2.2)
(1.1, 2.2)	2.1	0.1	0.21	(1.2, 2.41)
(1.2, 2.41)	2.2	0.1	0.22	(1.3, 2.63)
(1.3, 2.63)	2.3	0.1	0.23	(1.4, 2.86)
(1.4, 2.86)	2.4	0.1	0.24	(1.5, 3.1)

Euler's Method leads us to the approximation $f(1.5) \approx 3.1$.